Examples :→

1. Show that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

Proof :→ Let, $\varepsilon > 0$ be arbitrary.

Let, $f(x) = x\sin\frac{1}{x}$.

Then $|f(x)-0| = |x\sin\frac{1}{x} - 0| = |x\sin\frac{1}{x}|$

$$= |x|\cdot|\sin\tfrac{1}{x}|$$

$$\le |x| \quad [\because |\sin\tfrac{1}{x}| \le 1]$$

$$< \varepsilon \text{ whenever } |x-0| < \varepsilon.$$

Thus we see that if we choose $\delta = \varepsilon$, then $|x\sin\frac{1}{x} - 0| < \varepsilon$ whenever $|x-0| < \delta$.

Hence $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

2. By definition of limit of a function show that
$$\lim_{x\to\infty} \frac{3x-1}{3-4x} = -\frac{3}{4}.$$

Proof :→ Let, $\varepsilon > 0$ be arbitrary.

We have to prove that $\left|\frac{3x-1}{3-4x} - \left(-\frac{3}{4}\right)\right| < \varepsilon$ whenever $x > G$, where G is a however large positive number.

Now $\left|\frac{3x-1}{3-4x} - \left(-\frac{3}{4}\right)\right| = \left|\frac{3x-1}{3-4x} + \frac{3}{4}\right|$

$$= \left|\frac{12x - 4 + 9 - 12x}{4(3-4x)}\right|$$

$$= \left|\frac{5}{4}\cdot\frac{1}{3-4x}\right| < \varepsilon \text{ whenever}$$

$$\frac{5}{4}\left|\frac{1}{3-4x}\right| < \varepsilon.$$

$$\Rightarrow \frac{1}{|3-4x|} < \frac{4\varepsilon}{5}$$

$$\Rightarrow |3-4x| > \frac{5}{4\varepsilon}$$

i.e. $3-4x > \frac{5}{4\varepsilon}$ or $3-4x < -\frac{5}{4\varepsilon}$.

$\therefore \left|\frac{3x-1}{3-4x} - \left(-\frac{3}{4}\right)\right| < \varepsilon$ whenever $3-4x < -\frac{5}{4\varepsilon}$.

i.e. if $4x > 3 + \frac{5}{4\varepsilon}$, i.e. if $x > \frac{12\varepsilon+5}{16\varepsilon} = G$, say.

Hence we find G.

So $\lim_{x \to \infty} \frac{3x-1}{3-4x} = -\frac{3}{4}$.

3. From definition of limit show that

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$$

Proof :→ Let $\varepsilon > 0$ be arbitrary.

Now we have to find a $\delta > 0$ s.t.

$\left| \frac{x^2-1}{x-1} - 2 \right| < \varepsilon$ whenever $|x-1| < \delta$.

Now $\left| \frac{x^2-1}{x-1} - 2 \right| = \left| \frac{x^2-1-2(x-1)}{x-1} \right|$

$= \left| \frac{x^2-2x+1}{x-1} \right|$

$= \left| \frac{(x-1)^2}{x-1} \right|$

$= |x-1| < \varepsilon$ whenever $|x-1| < \varepsilon$

i.e. $\left| \frac{x^2-1}{x-1} - 2 \right| < \varepsilon$, whenever $|x-1| < \delta$, where $\delta = \varepsilon$.

So $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$.

4. Does the limit $\lim_{x \to 1} [x+2]$ exist?

Solution :→ $[x+2] = \begin{cases} 2, & \text{if } 0 \le x < 1 \\ 3, & \text{if } 1 \le x < 2 \end{cases}$

$\therefore \lim_{x \to 1+} [x+2] = \lim_{x \to 1+} 3 = 3$.

And $\lim_{x \to 1-} [x+2] = \lim_{x \to 1-} 2 = 2$.

Since $\lim_{x \to 1+} [x+2] \neq \lim_{x \to 1-} [x+2]$, so $\lim_{x \to 1} [x+2]$ does not exists.

Exercises :→

1. Let, $f(x) = \begin{cases} x+1, & \text{if } x>0 \\ 0, & \text{if } x=0 \\ 2-x, & \text{if } x<0 \end{cases}$. Find $\lim_{x\to 0} f(x)$.

2. A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^2+ax, & \text{if } 0 \le x < 1 \\ 3-bx^2, & \text{if } 1 \le x \le 2. \end{cases}$$

If $\lim_{x\to 1} f(x) = 4$, find a and b.

3. Does $\lim_{x\to 1} \sqrt{x-1}$ exist? Give reasons.

4. From definition of limit of a function show that:

i) $\lim_{x\to 2} \frac{2x^2-8}{x-2} = 8$

ii) $\lim_{x\to 0} x\cos\frac{1}{x} = 0$

iii) $\lim_{x\to \infty} \frac{2x+3}{4x-1} = \frac{1}{2}$.

iv) $\lim_{x\to 1} \frac{1}{(1-x)^2} = \infty$

5. Does $\lim_{x\to 0} \frac{|x|}{x}$ exist?

6. Does $\lim_{x\to 1} [x]$ exist? If yes, find the limit.

7. Does the limit $\lim_{x\to 0} \frac{\sin[x]}{[x]}$ exist?

8. $f(x) = 1-|x|$, $-1 \le x \le 1$.

Find $\lim_{x\to 0+} f(x)$, $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0} f(x)$.